Latin Squares for Elementary and Middle Grades

Yul Inn
Fun Math Club
email: Yul.Inn@FunMathClub.com
web: www.FunMathClub.com

Abstract: A Latin square is a simple combinatorial object that arises in many areas of discrete mathematics. We explore how Latin squares can be used as an aid for teaching elementary and middle school mathematics skills from problem solving and arithmetic to recognizing patterns and symmetry.

A Latin square is a combinatorial object whose definition is based on very simple concepts and logical conditions. Its simplicity makes it accessible to elementary grade students. Activities involving Latin squares can be used to exercise arithmetic and mathematical reasoning skills, and to provide examples of symmetry, combinations and permutations, and interactions between mathematics and art.

What is a Latin Square?

Before formally defining a Latin square, let's consider the following problem illustrating an application of Latin squares. Suppose a farmer has 3 varieties of tomatoes and 3 types of fertilizer. His farmland is divided into 3 fields, each with a different soil type. He would like to determine which variety grows most efficiently in which field and with which type of fertilizer. As a test he might want to make sure he tries each triple of variety, fertilizer, and field. But this would result in many test cases (27). If he would be satisfied ensuring that each pair of variables is compared, he can do much better. Here's how:

Assume that the field, fertilizers, and tomato varieties are each numbered 1, 2, and 3. Plant the three tomato varieties in each of the fields and fertilize according to the table:

Field \ Fertilizer	1	2	3
1	1	2	3
2	2	3	1
3	3	1	2

For example, in field 2 grow tomato variety 2 using fertilizer 1, variety 3 using fertilizer 2, and variety 1 using fertilizer 3.

This solution has only 9 test cases yet it has the property that:

1. Each variety is grown in each field.
2. Each fertilizer is used in each field.
3. Each tomato is fertilized with each fertilizer.

This arrangement is an example of a Latin square. More formally, a *Latin square* is an n x n matrix whose elements are one of n symbols such that each symbol occurs exactly once in each row and column. The size n is called the *order* of the Latin square. The representation of a Latin square is normally a square grid of *cells*, each cell containing a symbol.

For example, here is a 4x4 Latin square on the symbols α, β, χ, and δ.

α	β	χ	δ
χ	δ	α	β
δ	χ	β	α
β	α	δ	χ

The symbols in a Latin square are arbitrary. Symbols can be numbers, letters, geometric shapes, and colors. In the classroom, crayons, stickers, or other objects may be used to create, or to serve as, the symbols.

In most examples and worksheets, we will use the first n natural numbers or the first n letters for the symbols in Latin squares of order n.

Latin Square Puzzles

A *partial* Latin square is an n x n grid of cells in which some cells are filled with symbols, and no symbol occurs more than once in any row or column. It may be possible to *complete*, i.e. fill the remaining cells of, a partial Latin square with symbols so that the completed grid is a Latin square. Here is an example:

A

1	**2**	
	1	

B

1	**2**	3
2	3	1
3	**1**	2

Partial Latin square A can be completed to form Latin square B.

Not all partial Latin squares can be completed. Here is one example:

1	**2**	
		3

The upper right cell cannot be filled, so this partial Latin square cannot be completed.

Problems to complete a partial Latin square can make good exercises in logical reasoning. Problems of this type can range from very easy to arbitrarily complex, depending on the size of the square and the number of symbols filled in.

Solving Partial Latin Square Problems

There are three basic types of reasoning that can aid in solving these problems:

Row-forced or Column-forced entries:

> If all but one cell of a row or column is filled, the symbol in remaining cell must be the one remaining symbol.

Row-and-column-forced entries:

> For a given cell, if all symbols except one appear in either the row of the cell or the column of the cell, then the one remaining symbol must be in the cell.

Last-cell-forced entries:

> If a symbol occurs in all but one of the rows (and columns) its last occurrence is forced to be in the row and column in which it does not yet occur.

Note: the first type of reasoning is actually a special case of the second, but it is so much easier to identify the first case that it makes sense to call it out separately.

In the example below (where the symbols are 1,2,3,4), the cell with the X must be 4 by the row-forced reasoning and the cell with the Y must be 3 column-forced reasoning. The cell marked by Z must be 4 using the row-and-column-forced reasoning. The cell marked W must be 2 by the last-cell-forced reasoning.

2	X	3	1
1		Z	W
Y	2		
4		2	

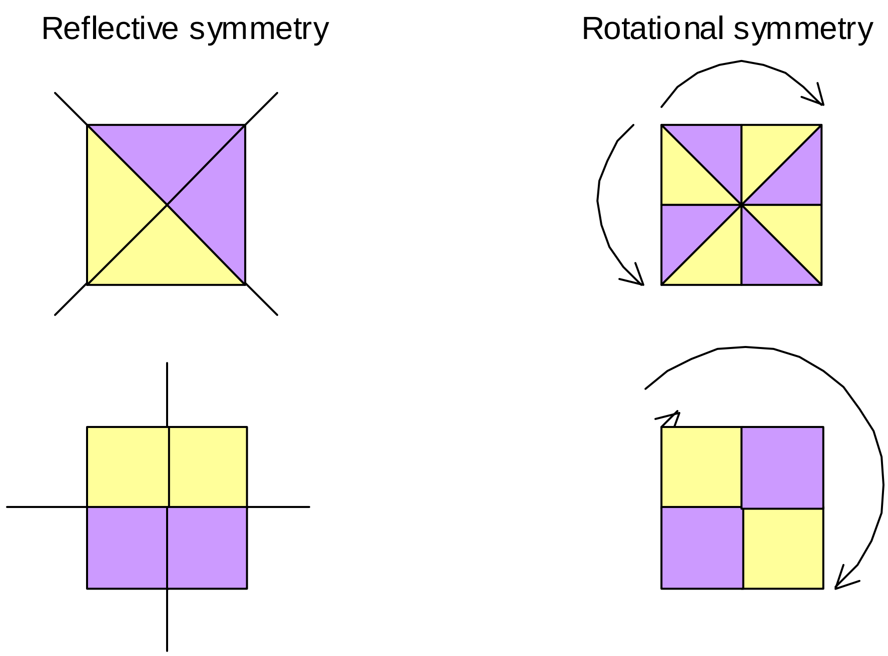
Worksheets

The “Latin Square Puzzles” worksheet has a collection of problems to complete partial Latin squares.

Exploring Symmetry with Latin Squares

A square has four reflective and four rotational symmetries. Latin squares have symmetries determined by the symbols in its cells. It takes only a little bit of experimentation to see that a Latin square cannot have a reflective symmetry along either the vertical or horizontal axis.

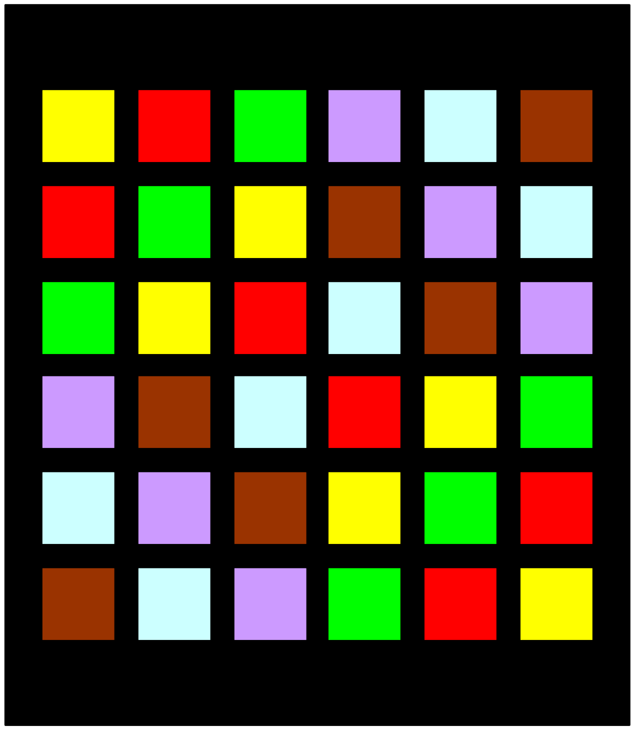
What other symmetries are possible?



Worksheets

The “Latin Square Symmetry” worksheet includes some partial Latin squares that can be completed to form symmetric and doubly symmetric Latin squares.

Colorful Latin Squares



Latin Square Mosaics

Using colored squares as the symbols for a Latin Square produces some nice mosaic designs. To make a Latin square mosaic of order 6, cut out six 1”x1” squares of 6 different colors of construction paper. Paste the squares in a Latin square pattern on a piece of black (or some other contrasting color) construction paper.

Orthogonal Latin Squares

A pair of Latin squares of order n is *orthogonal* if the n^2 ordered pairs comprised of elements in corresponding cells of the two squares are all distinct. The diagram below illustrates combing two Latin squares. In the square on the right it can be verified that all order pairs (written as two digit numbers) are distinct.

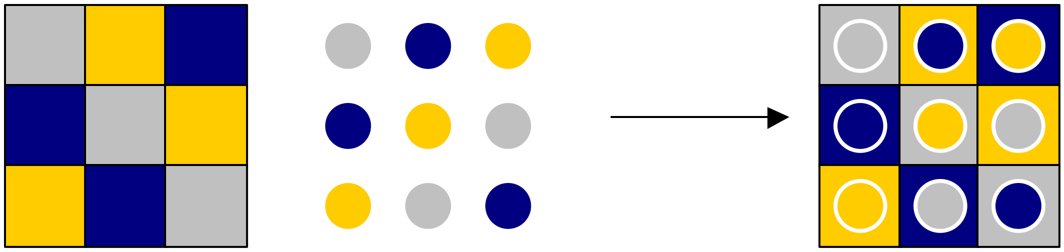
1	2	3
3	1	2
2	3	1

1	3	2
3	2	1
2	1	3

→

11	23	32
33	12	21
22	21	13

Superimposing orthogonal Latin squares where symbols are different colors and cells of different Latin squares are different shapes results in some attractive designs.



History

Euler was probably the first mathematician to explore Latin squares. In 1779 he tried to solve the well-known 36-officer problem:

> 36 officers are of 6 different ranks and from 6 different regiments. One officer of each rank comes from each of the regiments. Is it possible to arrange them in a 6x6 array so that each row and column has one officer of each rank and one from each regiment?

In the terminology of Latin squares, this problem asks if there exists two orthogonal Latin squares of order 6. Euler conjectured that two such squares do not exist. In fact he conjectured that no pair of orthogonal squares exist for any order of the form 4n+2. In 1900 his conjecture proved correct for order 6 squares but in 1959 it was shown that he was incorrect for all orders greater than 6!

Worksheets

In the classroom, using the term “matched” instead of orthogonal is less imposing for younger students. The “Are These Squares Matched?” worksheet includes some exercises for identifying orthogonal (matched) Latin squares. The two “Latin Square Art” worksheets have two different orthogonal Latin square designs, one of order 4 and one of order 5.

Modular Arithmetic and Latin Squares

Computations of hours on a clock are examples modular arithmetic. On our 12-hour clock the arithmetic is that of Z_{12}, the integers modulo 12. The addition and multiplication tables of modular arithmetic can provide examples of Latin squares. In the classroom, these can be introduced using clocks from alien worlds where days are not 24 hours long. For example, on the alien planets ModFive, ModSix, or ModSeven, days are 10, 12, or 14 hours, respectively. Here the clocks have only 5,6, or 7 hours.

Modular addition is easy to understand using the clock as a counting aid. There are two ways to treat multiplication:

1. As repeated addition. Add the multiplicand the number of times specified by the multiplier using modular addition.
2. As multiplication followed by reduction. Perform the multiplication using integer arithmetic, and reduce the result by the modulo. To perform the reduction count around the clock starting at 1 to determine which “hour” the count ends on. Students

may begin to recognize the reduced value as the remainder upon division by the modulo.

Worksheets

The “Clock Arithmetic” worksheet has some practice mod 12 addition problems using our 12-hour clock. The “ModFive Addition and Multiplication”, “ModSix Clock Arithmetic”, and “ModSeven Clock Arithmetic” have exercises that explore the modular arithmetic addition and multiplication tables and their relationship to Latin squares.

Although the worksheets have questions to answer, other questions to consider and discuss when using the worksheet include:

1. Why is the addition table Latin square symmetric?
2. Why is the multiplication table Latin square symmetric?
3. What numbers have Latin squares in their clock arithmetic multiplication tables? Why?

Permutations and Latin Squares

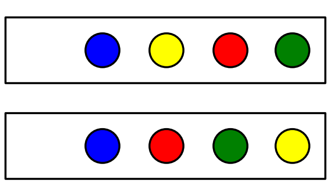
Latin squares can be viewed as a 2-dimensional permutation. Indeed, each row and column is a permutation of the symbols. Permutation bars can be used to explore permutations and Latin squares. A *permutation bar* is a rectangular strip with a permutation of a set of objects on it. A set of permutation bars contains one strip for each of the possible permutations of a given set of objects.

The optimal number of objects for permutation bars is four. This results in 24 bars. Using only three objects is possible, but the puzzles and games with the six bars are rather trivial. Using five or more objects results in too many bars.

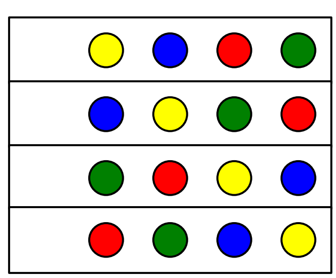
A set of permutation bars made from strips of poster board and colored dot stickers or other types of stickers serves as a simple manipulative from which to construct Latin square designs. (Note: when making permutation bars, it is helpful to make the strip asymmetric; otherwise, permutations may become duplicated by turning a bar end-for-end.)

The activities below are based on a set of 24 permutation bars.

Permutation bars



Latin square with Permutation bars



Partial Permutation Latin Squares

Can you complete every partial Latin square with one, two, or three permutation bars into a Latin square?

Latin Square Permutation Bar Game

Players shuffle and deal bars face down to each player. The players' hands consist of the bars dealt to them. Players may look at their own hands. The game consists of a number of rounds. In a round each player take turns placing a bar from their hand on the table. Players alternate going first in a round. After the first bar has been played, subsequent bars are laid next to those previously played in the round. At each play, the set of bars played must form partial Latin squares (or at the end a completed Latin square). A round consists of the four plays. A player who cannot play a bar loses.

Variations:

1. Allow bars to be played in either orientation.
2. Allow up to six (partial) squares to be on the table at one time (only one round in this case). A player may play on any of the squares or start a new one if there are less than six.

Permutation Bar Puzzle

Can you partition the bars into six sets of four so that each set can forms a Latin square? (This would show that ties are possible in the game.)